

Receding Horizon Control with Incomplete Observations

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1 Introduction.

We consider the optimal control problem of minimizing the performance index subject to a nonlinear control system:

$$(1.1) \quad \min \int_0^{T_\infty} f^0(x(t), u(t)) dt$$

where

$$(1.2) \quad \frac{d}{dt}x(t) = f(x(t)) + Bu(t), \quad \text{for } t > 0,$$

$$(1.3) \quad x(0) = x_0,$$

for some $T_\infty \in (0, \infty]$ and $x_0 \in R^n$. We refer to $x(\cdot)$ and $u(\cdot)$ as state and control functions with $x(t) \in R^n$ and $u(t) \in R^m$. Under appropriate conditions (1.1)–(1.3) admits a solution which satisfies the minimum principle

$$(1.4) \quad \begin{cases} \frac{d}{dt}x(t) = H_p(x(t), u(t), p(t)), & x(0) = x_0, \\ \frac{d}{dt}p(t) = -H_x(x(t), u(t), p(t)), & p(T_\infty) = 0, \\ u(t) = \arg \min_{u \in R^m} H(x(t), u, p(t)), \end{cases}$$

where H is the Hamiltonian defined by $H(x, u, p) = f^0(x, u) + p \cdot f(x, u)$. The coupled system of two-point boundary value problems with initial condition for the primal equation and terminal condition for the adjoint equation represents a significant challenge for numerical computations in case the dimension n of the state or the time horizon T_∞ are large. It has therefore been the focus of many research efforts. An alternative approach consists in constructing the feedback solution based on Bellman's dynamic programming principle. Again, due to computational costs, this is only tractable for very limited examples.

One of the possibilities to overcome these difficulties is given by time-domain decomposition based on receding horizon formulations [ABQRW, GPM]. Receding horizon techniques have proved to be effective numerically both for optimal control problems governed by ordinary (e.g. [CA, JYH, K, MM, PND, SMR]) and by partial differential equations, e.g. in the form

of the instantaneous control technique for problems in fluid mechanics [B, CHK, CTMC, HV].

To briefly explain the strategy let $0 = T_0 < T_1 \dots < T_\infty$ describe a grid on $[0, T_\infty)$ and let $T \geq \max\{T_{i+1} - T_i : i = 0, \dots\}$. If $T > T_{i+1} - T_i$ we have overlapping domains. The receding horizon optimal control problem involves the successive finite horizon optimal control on $[T_i, T_i + T]$:

$$(1.5) \quad \min \int_{T_i}^{T_i+T} f^0(x(t), u(t)) dt + G(x(T_i + T)),$$

subject to

$$(1.6) \quad \frac{d}{dt}x(t) = f(x(t)) + Bu(t), \quad t \geq T_i,$$

$$(1.7) \quad x(T_i) = x^*(T_i) \text{ if } i \geq 1 \text{ and } x(0) = x_0 \text{ for } i = 0,$$

where x^* is the solution to the auxiliary problem on $[T_{i-1}, T_{i-1} + T]$.

The solution on $[0, T_\infty)$ is obtained by concatenation of the solutions on $[T_i, T_{i+1}]$ for $i = 0, \dots$. If the terminal cost G is chosen as a control Liapunov function, then the asymptotic stability and the performance estimate of the receding horizon synthesis are established in [IK1, IK2].

If $x(T_i)$ is observed, then the receding horizon control technique is a state feedback method since the control on $[T_i, T_{i+1}]$ is determined as a function of the state $x^*(T_i)$. If the optimal pair $(x^*(t), u^*(t))$, $t \in [T_i, T_i + T]$ is shifted by $-T_i$ it satisfies the two point boundary value problem (1.4) on the interval $[0, T]$ with the terminal condition $p(T) = G_x(x(T))$ and initial condition $x(0) = x^*(T_i)$.

In this paper we address the state estimator problem for the receding horizon technique. We shall also allow for additive noise in the system dynamics as well as in the observation process and we admit uncertainty in the initial condition. The nonlinear autonomous control system with additive unmodeled disturbance is given by

$$(1.8) \quad \begin{cases} \frac{d}{dt}x(t) = f(x(t)) + Bu(t) + d(t), & \text{for } t > 0, \\ x(0) = x_0, \end{cases}$$

where $d(t)$ is an unknown disturbance process. The observation process providing partial observations $y(t) \in R^p$ of the state $x(t)$ is assumed to be of

the form

$$(1.9) \quad y(t) = Cx(t) + n(t),$$

where $C \in \mathbb{R}^{p \times n}$ and $n(t)$ is a measurement noise process. The output feedback law will utilize the open loop optimal control u^* with associated optimal state x^* on the interval $[T_i, T_{i+1}]$ computed from (1.5), (1.6) with initial condition (1.8) for $i \geq 1$ and

$$x(0) = x_0 + \eta_0,$$

for $i = 0$, where η_0 denotes the uncertainty in the initial condition.

The output feedback law is chosen to be of the form

$$(1.10) \quad u(t) = u^*(t) - B^*\Pi(w(t) - x^*(t)), \quad t \in [T_i, T_{i+1}),$$

where w denotes the state of the compensator. The construction of the feedback gain $B^*\Pi$ will be specified below. Suggested from LQG design the compensator is based on (1.10) together with the state estimator dynamics of the form

$$(1.11) \quad \frac{dw}{dt} = f'(\bar{x})(w(t) - x^*(t)) + f(x^*(t)) + Bu(t) + \Sigma C^*(y(t) - Cw(t)),$$

$$(1.12) \quad w(T_i) = w(T_i^-) \text{ for } i = 1, 2, \dots, \text{ and } w(0) = x_0 + \eta,$$

for the state estimator $w(t)$, where $w(T_i^-)$ denotes the value of the compensator on $[T_{i-1}, T_{i-1} + T]$ at T_i . Thus we linearize (1.1) at a reference point \bar{x} and construct the control gain $B^*\Pi$ and the filter gain ΣC^* by two LQG Riccati equations which we specify in the following section. The reference state \bar{x} is selected on the basis of the optimal pair (u^*, x^*) on $[T_i, T_i + T]$, for example

$$(1.13) \quad \bar{x} = \frac{1}{T} \int_{T_i}^{T_i+T} x^*(t) dt \text{ or } \bar{x} = x^*(T_i + T).$$

The feedback synthesis (1.10) performs tracking of (1.8) to the optimal pair (u^*, x^*) on $[T_i, T_i + T]$ under the uncertainty of the initial condition, observation noise and an additive disturbance in the system dynamics. Concerning the initial conditions for the open loop reference solution x^* and the

state estimator w we can will also consider the situation, where the state is observable with uncertainty η_i at the grid points T_i , i.e. we consider

$$(1.14) \quad x^*(T_i) = w(T_i) = x(T_i, u) + \eta_i, \text{ for } i = 1, 2, \dots,$$

where $x(T_i, u)$ is the value of the controlled trajectory to (1.8).

The stability analysis and the performance estimate are carried out in Section 2. The asymptotic behavior of the overall closed-loop system (1.8), with feedback control given by (1.10) and (1.11) is discussed in Section 3.

2 LQG Design

In this section we describe the construction of the feedback and filters gains for (1.10) and (1.11). Subsequently we establish the stability and the performance estimate for the compensator dynamics (1.10), (1.11) based on the LQG design on a single time-horizon $[0, T]$. The iterative procedure on the sequence of time horizons $[T_i, T_{i+1}]$ will be considered in Section 3. The Jacobian of f at $y \in \mathbb{R}^n$ will be denoted by $A(x) = f'(x)$ and in particular

$$A = f'(\bar{x}).$$

We assume that

$$(2.1) \quad (A, B, C) \text{ is stabilizable and detectable .}$$

Consider the controlled linear equation , which results from linearizing f at \bar{x} :

$$(2.2) \quad \begin{cases} \frac{d}{dt}(\hat{x}(t) - x^*(t)) = A(\hat{x}(t) - x^*(t)) + B(u(t) - u^*(t)) + d(t), \\ \hat{x} = x_0 + \eta_0, \end{cases}$$

where $d(t)$ is the disturbance process. We shall further comment on the choice of the linear equation in Remark 2.1 below. Let $Q \in \mathbb{R}^{n \times n}$ denote a positive definite matrix and consider the tracking problem to the pair (u^*, x^*) :

$$(2.3) \quad \begin{cases} \min \frac{1}{2} \int_0^\infty ((\hat{x}(t) - x^*(t))^T Q (\hat{x}(t) - x^*(t)) + |u(t) - u^*(t)|^2) dt \\ \text{subject to (2.2) over } u \in L^2(0, \infty; \mathbb{R}^m). \end{cases}$$

Let $V: \mathbb{R}^n \rightarrow \mathbb{R}^+$ denote the value functional associated to (2.3). It can be shown that

$$V(x) = \frac{1}{2}(x - x^*)^T \Pi (x - x^*) + v^T (x - x^*) + c,$$

where c is a constant, and the symmetric positive definite matrix $\Pi \in \mathbb{R}^{n \times n}$ and $v \in \mathbb{R}^n$ satisfy

$$(2.4) \quad \begin{cases} A^* \Pi + \Pi A - \Pi B B^* \Pi + Q = 0 \\ (A - B B^* \Pi)^* v + \Pi d = 0, \end{cases}$$

where we assume that the disturbance is constant in time. The optimal feedback loop control for (2.3) is given by

$$(2.5) \quad u = u^*(t) - B^*(\Pi(\hat{x}(t) - x^*(t)) + v).$$

Since we consider unknown, unmodeled disturbance $d = d(t)$ we do not include the feedforward input v in the feedback form (1.10). Turning to the estimator we use the (Kalman) filter gain ΣC^* based on the linear system (A, C) where $\Sigma \in \mathbb{R}^{n \times n}$ is the positive definite solution to

$$(2.6) \quad A \Sigma + \Sigma A^* - \Sigma C^* C \Sigma + R = 0,$$

with $R \in \mathbb{R}^{n \times n}$ a positive definite matrix. This results in the following equations for the compensator

$$(2.7) \quad \frac{dw}{dt} = A(\bar{x})(w(t) - x^*(t)) + f(x^*(t)) + Bu(t) + \Sigma C^*(y(t) - Cw(t)),$$

$$(2.8) \quad w(0) = x_0 + \eta_0,$$

and the associated feedback law

$$(2.9) \quad u(t) = u^*(t) - B^* \Pi (w(t) - x^*(t)).$$

Note that (2.1) guarantees the existence of Π and Σ with the specified properties. Moreover the spectra of $A - B B^* \Pi$ and $A - \Sigma C^* C$ are strictly contained in the left half of the complex plane. We henceforth assume that

$$d \in L^1_{loc}(0, \infty; \mathbb{R}^n), \text{ and } n \in L^1_{loc}(0, \infty; \mathbb{R}^p).$$

We further assume the existence of solutions x and w to (1.8) and (2.7), (2.8) on $[0, T]$ for every $u \in L^1(0, T; \mathbb{R}^m)$, where y is given by (1.9).

Proposition 2.1 Let $W(x) = \frac{1}{2}x^T \Pi x$. For $t \in [0, T]$ we have

$$\begin{aligned} \frac{d}{dt}W(x(t) - x^*(t)) &= -\frac{1}{2}(|B^*\Pi(x(t) - x^*(t))|^2 + (Q(x(t) - x^*(t)), x(t) - x^*(t))) \\ &\quad + (d(t) + r(x(t), x^*(t)), \Pi(x(t) - x^*(t))) + (B^*\Pi(x(t) - w(t)), B^*\Pi(x(t) - x^*(t))), \end{aligned}$$

where

$$r(x, x^*) = f(x) - f(x^*) - A(\bar{x})(x - x^*),$$

and x is the solution to (1.8) with u given in (2.9) and x^* is the solution to (1.6).

Proof. Using (1.6) and (1.8) we have

$$\begin{aligned} \frac{d}{dt}(x(t) - x^*(t)) &= A(x(t) - x^*(t)) - BB^*\Pi(x(t) - x^*(t)) \\ &\quad + BB^*\Pi(x(t) - w(t)) + d(t) + r(x(t), x^*(t)). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}W(x(t) - x^*(t)) &= ((A - BB^*\Pi)(x(t) - x^*(t)) + BB^*\Pi(x(t) - w(t)) \\ &\quad + d(t) + r(x(t), x^*(t)), \Pi(x(t) - x^*(t))) \end{aligned}$$

and the proposition follows from the fact that

$$((A - BB^*\Pi)x, \Pi x) = -\frac{1}{2}(|B^*\Pi x|^2 + (Qx, x)). \quad \square$$

Proposition 2.2 Let $\tilde{W}(z) = \frac{1}{2}z^T \Sigma^{-1}z$. For $t \in [0, T]$ we have

$$\begin{aligned} \frac{d}{dt}\tilde{W}(x(t) - w(t)) &= -\frac{1}{2}(|C(x(t) - w(t))|^2 + (R\Sigma^{-1}(x(t) - w(t)), \Sigma^{-1}(x(t) - w(t))) \\ &\quad + (r(x(t), x^*(t)) + d(t), \Sigma^{-1}(x(t) - w(t))) - (n(t), C(x(t) - w(t))), \end{aligned}$$

where x and w are the solutions to (1.8) and (2.7) for some $u \in L^2(0, T; \mathbb{R}^m)$.

Proof. From (1.8) and (2.7) we have

$$\frac{d}{dt}(x(t)-w(t)) = A(x(t)-w(t))+r(x(t), x^*(t))+d(t)-\Sigma C^*(C(x(t)-w(t))+n(t)),$$

and hence

$$\begin{aligned} \frac{d}{dt}\tilde{W}(x(t) - w(t)) &= ((A - \Sigma C^* C)(x(t) - w(t)) \\ &\quad + r(x(t), x^*) + d(t) - \Sigma C^* n(t), \Sigma^{-1}(x(t) - w(t))). \end{aligned}$$

Thus the proposition follows from the fact that

$$((A - \Sigma C^* C)z, \Sigma^{-1}z) = -\frac{1}{2}(|Cz|^2 + (R\Sigma^{-1}z, \Sigma^{-1}z)). \quad \square$$

To quantify the performance of the compensator we set

$$E(t) = (x(t) - x^*(t), \Pi(x(t) - x^*(t))) + (x(t) - w(t), \Sigma^{-1}(x(t) - w(t)))^{1/2}.$$

Note that $E(t) = (W(x(t) - x^*(t)) + \tilde{W}(x(t) - w(t)))^{1/2}$. We further introduce positive constants α_1 , α_2 and β_1 , β_2 such that

$$(2.10) \quad \alpha_1 \leq \Pi \leq \alpha_2, \quad \beta_1 \leq \Sigma^{-1} \leq \beta_2.$$

We shall require the following assumptions:

$$(2.11) \quad \begin{aligned} &\frac{1}{2}(|B^* \Pi x|^2 + (Qx, x) + |Cz|^2 + (R\Sigma^{-1}z, \Sigma^{-1}z)) \\ &\quad - (B^* \Pi \Sigma \Sigma^{-1}z, B^* \Pi x) \geq \omega (W(x) + \tilde{W}(z)) \text{ for all } x, z \in R^n, \end{aligned}$$

and

$$(2.12) \quad E(0) \leq \frac{\delta}{2} \text{ and } |x^*(t) - \bar{x}| \leq \delta \text{ on } [0, T] \text{ for some } \delta > 0.$$

Turning to the effect of the nonlinearity we observe that

$$r(x, x^*) = f(x) - f(x^*) - A(x^*)(x - x^*) + (A(x^*) - A(\bar{x}))(x - x^*).$$

We assume that there exist a constant L such that

$$(2.13) \quad |r(x, x^*)| \leq L (|x^* - \bar{x}| + |x - x^*|) |x - x^*| \text{ for all } x, x^* \in \mathcal{S}$$

where $\mathcal{S} = \{x \in R^n : |x - \bar{x}| \leq \delta(1 + \frac{1}{\sqrt{\alpha_1}})\}$. We shall further assume that

$$(2.14) \quad \tilde{\omega} = \omega - L\delta \frac{\sqrt{\alpha_1} + 1}{\alpha_1} \frac{\sqrt{\alpha_2} + \sqrt{\alpha_2 + \beta_2}}{2} > 0.$$

Let us define

$$r(t) = \sqrt{\alpha_2 + \beta_2} |d(t)| + \frac{1}{\sqrt{\beta_1}} |C| |n(t)|.$$

We require the following smallness condition on the noise processes:

$$(2.15) \quad \int_0^t e^{-\tilde{\omega}(t-s)} r(s) ds < \frac{\delta}{2} \quad \text{for all } t \in [0, T].$$

Theorem 2.1 *If (2.12) and the stability conditions (2.11), (2.13), (2.14) and the smallness condition on the noise processes (2.15) are satisfied, then*

$$(2.16) \quad E(t) \leq e^{-\tilde{\omega}t} E(0) + \int_0^t e^{-\tilde{\omega}(t-s)} r(s) dt,$$

holds for all $t \in [0, T]$.

Proof. Due to (2.12) there exists τ such that

$$E(\tau) \leq \delta \text{ on } [0, \tau].$$

Note that $\sqrt{\alpha_1} |x(t) - x^*(t)| \leq E(t)$ and therefore $x(t) \in \mathcal{S}$ for $t \in [0, \tau]$. Set $X(t) = |x(t) - x^*(t)|_\Pi$ and $Y(t) = |x(t) - w(t)|_{\Sigma^{-1}}$. Suppressing the dependence on t we obtain by (2.12), (2.13) that

$$\begin{aligned} & (r(x, x^*), \Pi(x - x^*)) + (r(x, x^*), \Sigma^{-1}(x - w)) \\ & \leq L\delta \frac{\sqrt{\alpha_1} + 1}{\alpha_1} (\sqrt{\alpha_2} X^2 + \sqrt{\beta_2} XY) \\ & \leq L\delta \frac{\sqrt{\alpha_1} + 1}{\alpha_1} \frac{\sqrt{\alpha_2} + \sqrt{\alpha_2 + \beta_2}}{2} (X^2 + Y^2), \text{ for } t \in [0, \tau]. \end{aligned}$$

Since

$$E(t) \frac{d}{dt} E(t) = \frac{d}{dt} (W(x(t) - x^*(t)) + \tilde{W}(x(t) - w(t))),$$

we find by Proposition 2.1, 2.2, with u given in (2.9), and (2.11),(2.13), (2.14) that

$$\begin{aligned} & \frac{d}{dt} (W(x - x^*) + \tilde{W}(x - w)) \\ & \leq -\omega(W(x - x^*) + \tilde{W}(x - w)) + (d + r(x, x^*), \Pi(x - x^*) + \Sigma^{-1}(x - w)) - (n, C(x - w)) \\ & \leq -\tilde{\omega}(W(x - x^*) + \tilde{W}(x - w)) + (d, \Pi(x - x^*)) + (d, \Sigma^{-1}(x - w)) - (n, C(x - w)) \\ & \leq -\tilde{\omega}(W(x - x^*) + \tilde{W}(x - w)) + r(t)((W(x - x^*) + \tilde{W}(x - w))^{1/2}). \end{aligned}$$

Consequently

$$\frac{d}{dt} E(t) \leq -\tilde{\omega}E(t) + r(t).$$

This implies (2.16) on $[0, \tau]$.

By (2.15) and continuity of $t \rightarrow \int_0^t e^{-\omega(t-s)} r(s) ds$ there exists $\alpha \in (0, 1)$ such that

$$\int_0^t e^{-\tilde{\omega}(t-s)} r(s) ds \leq \frac{\alpha\delta}{2} \text{ for all } t \in [0, T],$$

and consequently

$$(2.17) \quad E(t) < \frac{\delta}{2}(1 + \alpha) \text{ for all } t \in [0, \tau].$$

A continuation argument implies that (2.16) and (2.17) hold on $[0, T]$. In fact, if (2.17) holds on $[0, T]$, then (2.16) can be continued from $[0, \tau]$ to $[0, T]$. Assuming that (2.16) is not valid on $[0, T]$, let $\tilde{\tau}$ denote the smallest value in $(0, T)$ such that $E(\tilde{\tau}) = \frac{\delta}{2}(1 + \alpha)$. Then repeating the argument leading to (2.16) it can be shown that there exists $\epsilon > 0$ such that (2.16) holds in $[0, \tilde{\tau} + \epsilon]$. This implies $E(\tilde{\tau} + \epsilon) < \frac{\delta}{2}(1 + \alpha)$, which is a contradiction. Thus (2.17) holds on $[0, T]$. \square .

Remark 2.1 The linear equation (2.2) as a basis for the construction of the compensator is well suited for our purposes. Other choices are briefly indicated.

1. Linearizing (1.8) at \bar{x} results in

$$\frac{d}{dt}(\hat{x}(t) - \bar{x}) = f(\bar{x}) + A(\hat{x}(t) - \bar{x}) + B(u(t) - u^*(t)) + d(t).$$

The resulting Riccati synthesis is of the form (2.4) with d replaced by $d + f(\bar{x})$. Since d is unknown, $f(\bar{x})$ would remain as a bias term and necessitate to modify W to be $W(x) = \frac{1}{2}x^T \Pi x + x^T v$, with the bias v changing from one horizon to the next.

2. It is also possible to employ the time varying linearization

$$(2.18) \quad \begin{cases} \frac{d}{dt}(\hat{x}(t) - x^*(t)) = A(t)(\hat{x}(t) - x^*(t)) + B(u - u^*(t)) + \tilde{n}(t), \\ \hat{x} = x_0, \end{cases}$$

where $A(t) = A(x^*(t))$ and use the corresponding time-varying gains $B^* \Pi(t)$ and $\Sigma(t) C^*$ determined by

$$\frac{d}{dt} \Pi(t) + A(t)^* \Pi(t) + \Pi(t) A(t) - \Pi(t) B B^* \Pi(t) + Q = 0$$

and

$$\frac{d}{dt} \Sigma(t) - A(t) \Sigma(t) - \Sigma(t) A(t)^* + \Sigma(t) B B^* \Sigma(t) - R = 0.$$

One can adapt our analysis and establish an error estimate analogous to (2.16) assuming that $\omega > 0$ and L are independent of $t \in [0, T]$.

3 Asymptotic performance of closed-loop system

We will apply Theorem 2.1 repeatedly on the intervals $[T_i, T_i + T]$. Let us briefly recall the procedure. The open loop solution x^* to (1.5)–(1.7) is computed on $[T_i, T_{i+1}]$ and based on it \bar{x} is determined, see (1.13). To refer to as specific horizon we henceforth use \bar{x}_{T_i} for \bar{x} . This determines $A(\bar{x}_{T_i})$ and allows to compute $\Pi = \Pi(\bar{x}_{T_i})$ and $\Sigma = \Sigma(\bar{x}_{T_i})$ of the corresponding Riccati

equations. The compensator can then be defined on the basis of (1.10)–(1.12). To simplify the following discussion we assume that $T_{i+1} - T_i = T$ for all i . We shall assume that

$$(3.1) \quad E(0) \leq \frac{\delta}{2} \text{ and } |x^*(t) - \bar{x}| \leq \delta \text{ on } [T_i, T_{i+1}] \text{ for all } i = 0, \dots$$

and that (2.10) and (2.11) hold uniformly on all horizons $[T_i, T_{i+1}]$. In view of continuity of $x \rightarrow A(x)$, $x \rightarrow \Pi(x)$ and $x \rightarrow \Sigma(x)$ and the fact that the open loop control x^* is typically guaranteed to be continuous and bounded on $[0, \infty)$, see e.g. [IK1], these assumptions are natural ones. As \bar{x}_{T_i} changes from one horizon to the next, so does $\mathcal{S} = \mathcal{S}_{T_i}$ in (2.13). We assume that (2.13) holds uniformly for all horizons as well and that

$$(3.2) \quad \int_0^t e^{-\tilde{\omega}(t-s)} r(s + T_i) ds < \frac{\delta}{2} \text{ for } t \in [0, T] \text{ and all } i = 0, 1, \dots$$

Finally we require that

$$(3.3) \quad \int_0^T e^{-\tilde{\omega}(T-t)} r(T_i + t) dt \leq \frac{\delta}{2}(1 - e^{-\tilde{\omega}T}), \text{ for } i = 0, 1, \dots$$

These assumptions imply that

$$(3.4) \quad E(t) \leq e^{-\tilde{\omega}(t-T_i)} E(T_i) + \int_0^{t-T_i} e^{-\tilde{\omega}(t-T_i-s)} r(T_i + s) ds \text{ for } t \in [T_i, T_{i+1}],$$

for every i , provided that $E(T_i) \leq \frac{\delta}{2}$. Note that (3.3), (3.4) imply

$$E(T_{i+1}) \leq e^{-\tilde{\omega}T} E(T_i) + \frac{\delta}{2}(1 - e^{-\tilde{\omega}T}) \leq \frac{\delta}{2},$$

and hence by induction $E(T_i) \leq \frac{\delta}{2}$ for all i .

Theorem 3.1 *If (3.1), (3.1) hold and (2.10), (2.11) and (2.13) are satisfied uniformly, then*

$$E(t) \leq e^{-\tilde{\omega}(t-T_i)} E(T_i) + \int_0^{t-T_i} e^{-\tilde{\omega}(t-T_i-s)} r(T_i + s) ds$$

for $t \in [T_i, T_{i+1}]$, and all i . In particular this implies

$$E(t) \leq e^{-\omega t} E(0) + \int_0^t e^{-\omega(t-s)} r(s) ds \text{ for all } t > 0.$$

Remark 3.1 *If the initialization of x^* and w on each interval $[T_i, T_{i+1}]$, $i \geq 1$ is carried out according to (1.14) then the first conclusion of the theorem remains valid provided (3.3) is replaced by*

$$(|\eta_i|_\Sigma + |\eta_i|_\Pi)^{1/2} \leq \frac{\delta}{2} \text{ for all } i = 0, 1, \dots$$

Note that on each horizon $[T_i, T_{i+1}]$ we must have $|x^*(t) - \bar{x}| \leq \delta$ on $[0, T]$. This may necessitate to take T smaller than $T_{i+1} - T_i$ to ensure that $|x^*(t) - \bar{x}| \leq \delta$ on $[T_i, T_i + T]$. In this case, we can further partition the interval $[T_i, T_{i+1}]$ into subintervals and use consecutive linearization on each subinterval so that the condition is satisfied. The extreme case of this procedure results in the time-varying synthesis as in Remark 2.1.

Concerning the condition $|x^*(t) - \bar{x}| \leq \delta$ we can use an alternative approach based on the H^∞ Riccati equation. This will be discussed in the next section.

4 H^∞ Riccati Synthesis

We assume that there exists an attenuation bound $\gamma > 0$ such that

$$(4.1) \quad A^* \Pi + \Pi A - \Pi (B B^* - \frac{1}{\gamma} I) \Pi + \frac{1}{\gamma} I + Q = 0$$

has a positive definite solution Π and

$$(4.2) \quad A \Sigma + \Sigma A^* - \Sigma (C^* C - \frac{1}{\gamma} I) \Sigma + \frac{1}{\gamma} I + R = 0$$

has a positive definite solution Σ . These Riccati equations are similar to those used in the equivalence between H^∞ controllers and linear quadratic zero-some differential games where in our case y and w are the two players [BB]. In the following proposition x and w denote the solutions to (1.8) and (2.7) with u given in (2.9) and x^* is the solution to (1.6), (1.8).

Proposition 4.1 *For $t \in [0, T]$ we have*

$$\begin{aligned} & \frac{d}{dt} W(x - x^*) \\ &= -\frac{1}{2} (|B^* \Pi(x - x^*)|^2 + (Q(x - x^*), x - x^*) + \frac{1}{\gamma} |\Pi(x - x^*)|^2 + \frac{1}{\gamma} |x - x^*|^2) \\ & \quad + (r(x, x^*) + d, \Pi(x - x^*)) + (B^* \Pi(x - w), B^* \Pi(x - x^*)), \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt} \tilde{W}(x - w) \\ &= -\frac{1}{2}(|C(x - w)|^2 + (R\Sigma^{-1}(x - w), \Sigma^{-1}(x - w)) + \frac{1}{\gamma}|x - w|^2 + \frac{1}{\gamma}|\Sigma^{-1}(x - w)|^2) \\ & \quad + (r(x, x^*) + d, \Sigma^{-1}(x - w)) - (n, C(x - w)), \end{aligned}$$

where

$$r(x, x^*) = f(x) - f(x^*) - A(\bar{x})(x - x^*),$$

and the dependence of the variables on t is suppressed.

Proof. The Proposition follows from the proofs of Propositions 2.1 and 2.2

observing that

$$((A - BB^*\Pi)x, \Pi x) = -\frac{1}{2}(|B^*\Pi x|^2 + (Qx, x) + \frac{1}{\gamma}|x|^2 + \frac{1}{\gamma}|\Pi x|^2),$$

and

$$((A - \Sigma C^*C)z, \Sigma^{-1}z) = -\frac{1}{2}(|Cz|^2 + (R\Sigma^{-1}z, \Sigma^{-1}z) + \frac{1}{\gamma}|z|^2 + \frac{1}{\gamma}|\Sigma^{-1}z|^2). \quad \square$$

Since

$$r(x(t), x^*(t)) = \left(\int_0^1 A(x^*(t) + \theta(x(t) - x^*(t))) d\theta - A \right) (x(t) - x^*(t)),$$

we can observe that the assumption

$$(4.3) \quad \left\| \int_0^1 A(x^*(t) + \theta(x(t) - x^*(t))) d\theta - A \right\| \leq \frac{1}{\sqrt{2}\gamma}$$

implies

$$(4.4) \quad (r(x(t), x^*(t)), \Pi(x(t) - x^*(t))) \leq \frac{1}{2\gamma}(|\Pi(x(t) - x^*(t))|^2 + \frac{1}{2}|x(t) - x^*(t)|^2).$$

and

$$(4.5) \quad (r(x(t), x^*(t)), \Sigma^{-1}(x(t) - x^*(t))) \leq \frac{1}{2\gamma}(|\Sigma^{-1}(x(t) - w(t))|^2 + \frac{1}{2}|x(t) - x^*(t)|^2).$$

Using (2.11) and (4.4), (4.5) we find

$$\begin{aligned}
& \frac{d}{dt}(W(x - x^*) + \tilde{W}(x - w)) \\
& \leq -\omega(W(x - x^*) + \tilde{W}(x - w)) + (d + r(x, x^*), \Pi(x - x^*) + \Sigma^{-1}(x - w)) - (n, C(x - w)) \\
& \quad - \frac{1}{2\gamma}(|\Pi(x - x^*)|^2 + |x - x^*|^2 + |x - w|^2 + |\Sigma^{-1}(x - w)|^2) \\
& \leq -\omega(W(x - x^*) + \tilde{W}(x - w)) + r(t)(W(x - x^*) + \tilde{W}(x - w))^{1/2} \\
& \quad + \frac{1}{2\gamma}(|\Pi(x - x^*)|^2 + |x - x^*|^2 + |\Sigma^{-1}(x - w)|^2) \\
& \quad - \frac{1}{2\gamma}(|\Pi(x - x^*)|^2 + |x - x^*|^2 + |x - w|^2 + |\Sigma^{-1}(x - w)|^2),
\end{aligned}$$

where $r(t)$ was defined below (2.14). Hence it follows that

$$\frac{d}{dt}(W(x - x^*) + \tilde{W}(x - w)) \leq -\omega(W(x - x^*) + \tilde{W}(x - w)) + r(t)((W(x - x^*) + \tilde{W}(x - w)))^{\frac{1}{2}},$$

and therefore

$$\frac{d}{dt}E(t) \leq -\omega E(t) + r(t).$$

We can summarize the above developments in the following result.

Theorem 4.1 *If (4.1) and (4.2) admit solutions for $\gamma > 0$ and (2.10), (2.11), (4.3) hold on $[0, T]$, then*

$$(4.6) \quad E(t) \leq e^{-\omega t} E(0) + \int_0^t e^{-\omega(t-s)} r(s) ds \text{ for } t \in [0, T].$$

If the assumptions hold uniformly on all intervals, then (4.6) holds for all $t \in [0, \infty)$.

5 Numerical examples

We validate the proposed approach by means of a class of optimal control problems for the Burgers equation:

$$(5.1) \quad \min \frac{1}{2} \int_0^{T_\infty} |y(t) - z(t)|_{L^2}^2 dt + \frac{\sigma}{2} \int_0^{T_\infty} |u(t)|_{L^2(\tilde{\Omega})}^2 dt$$

subject to $u \in L^2(\tilde{\Omega})$ and

$$(5.2) \quad \begin{cases} \frac{d}{dt} y(t) = \nu \Delta y(t) - y \cdot y_x + Bu(t) + d(t), & t > 0 \\ y(t, 0) = y(t, 1) = 0, & t > 0 \\ y(0, x) = \varphi(x), & x \in (0, 1), \end{cases}$$

where $y(t) = y(t, x)$, $x \in \Omega = (0, 1)$, and ν and σ are positive constants. Further $\tilde{\Omega} \subset \Omega$, $L^2 = L^2(\Omega)$, $\varphi \in L^2(\Omega)$, and B is the extension-by-zero-operator from $\tilde{\Omega}$ to Ω . Finally d represents noise to the system and the data are supposed to be of the form

$$(5.3) \quad \hat{y}(t) = C_{\hat{\Omega}} y(t) + n(t),$$

where C is the restriction operator from Ω to $\hat{\Omega} \subset \Omega$ and n stands for noise in the data. We further need to specify the operators G for the terminal weight in the receding horizon cost (1.5), the tracking weight Q in (2.3) and the operator R in the Kalman filter equation. We shall set

$$(5.4) \quad R = r I \quad \text{and} \quad Q = q I.$$

After several tests G was taken to be 0 for stabilization problems ($z = 0$) and as a scalar multiple of the solution of the algebraic Riccati equation (2.4) with $A = -\nu \Delta$ for tracking problems with $z \neq 0$. In this way G does not depend on the specific receding horizon level and can be computed during the initialization phase. Choosing Q instead as a multiple of the identity implies longer computing times and less favorable tracking properties.

The spatial discretization was done on the basis of linear finite elements with a Galerkin scheme applied to (5.2). The ordinary differential equations resulting from (5.2) and (2.2), (2.7) were solved with an implicit Euler scheme, while resolving the nonlinearities by Newton's method. The resulting linear systems were solved by inexact GMRES iterations. Unless specified otherwise we took

$$\nu = .001, \quad dt = .05, \quad dx = .025,$$

and for the receding horizon $T = .5$. Further, unless quoted otherwise we chose $\Omega = \tilde{\Omega} = \hat{\Omega}$, $q = 10^{-5}$ and $r = 10^3$. For Example 1 we took $\sigma =$

.0175 and for Example 2 we calculated with $\sigma = 10^{-3}$. The MATLAB-routine CARE we used to solve the algebraic Riccati equations. Below J_1 the tracking part of the cost in (5.1). Noise was simulated by choosing uniformly distributed random numbers in the interval $[-\delta, \delta]$. The initial condition for (5.2) on receding horizon intervals with $i \geq 2$ was chosen as the state of the estimator at time T_i .

Example 5.1 (*Stabilization*) Here we chose

$$\varphi(x) = \begin{cases} 5 \sin 2\pi x & \text{on } (0, \frac{1}{2}] \\ 0 & \text{on } (\frac{1}{2}, 1) \end{cases}$$

and $z = 0$. Further $T^\infty = 5$, $T = 0.5$, so that we have 10 receding horizon intervals. The uncontrolled solution is depicted in Figure 1.1. For Figures 1.2 - 1.4 we chose $d \neq 0$ and $n = 0$, with noise level $\delta = 10$ for d . The result for Figure 1.2 was produced by computing the open loop solution to (5.1) - (5.2), with $d = 0$ and using the optimal solution u^* thus obtained in a solve for the noisy Burgers equation. Figure 1.3 is obtained with the state estimator procedure introduced in section 1, with $\hat{\Omega} = \tilde{\Omega} = \Omega$, and for Figure 1.4 we chose $\tilde{\Omega} = (0, \frac{1}{2})$ and $\hat{\Omega} = (\frac{1}{2}, 1)$. It should be noted that the hyperbolic nature of the Burgers equation is such that information moves from smaller to larger x -values. For Figures 1.5 and 1.6 again $\tilde{\Omega} = (0, \frac{1}{2})$ and $\hat{\Omega} = (\frac{1}{2}, 1)$ but now $d = 0$ and $n \neq 0$, with noise level $\delta = 10$. In addition, for Figure 1.6 noise was added to the initial condition ($\delta = 5$). The run-time depends slightly on the random numbers that enter in d or n . Typical run-times for Figures 1.3 - 1.6 are 16 - 17 units, whereas the run-time for Figure 1.2, which uses the open loop solution from the complete interval $[0, T^\infty]$, is 181 units. The tracking value for Figure 1.2 is $\hat{J} = .54$, whereas it is only $\hat{J} = .30$ for Figure 1.3. In Table 1 we give the tracking values \hat{J}_- for the results of Figures 1.2 - 1.6 on $[.5, 5]$ i.e. for the receding horizon intervals 2-10.

Fig 1.2	Fig 1.3	Fig 1.4	Fig 1.5	Fig 1.6
.17-.5	.0018	.087	.072	.1-.5

Table 1: \hat{J}_-

We ran all tests with five randomly chosen initializations of the random number generator. In cases where the results depend significantly on the set of random numbers describing the noise we indicate the range of obtained values for \hat{J}_- in Table 1.

We can conclude that the receding horizon strategy combined with the dynamic observer introduced in section 1 attenuates noise significantly better than the open loop controlled system. The computing times for the receding horizon strategy are much shorter than for open loop solution on the complete time interval. Finally the strategy can cope with partial observations.

Example 5.2 (*Tracking*) Here we set

$$\varphi(x) = \begin{cases} 1 & \text{on } (0, \frac{1}{2}] \\ 0 & \text{on } (\frac{1}{2}, 1), \end{cases}$$

and z is the characteristic function of the set $\{(t, x): 2.5 - 5x < t < 5 - 5x, x \in (0, 1)\}$, see Figure 2.1. Note that the uncontrolled solution of the Burgers equation would transport the jump at $x = \frac{1}{2}$ towards increasing x as t increases, while decreasing its height. Also, the desired state z moves into the opposite direction from the characteristics of the Burgers equations. Thus this example can be considered as a challenging one. For Figures 2.2 and 2.3 we have $\Omega = \tilde{\Omega} = \hat{\Omega}$, whereas for Figures 2.4 - 2.6 $\tilde{\Omega} = \Omega$, and $\hat{\Omega} = (1/4, 1)$. For Figures 2.2 - 2.4 noise with noise-level $\delta = 5$ enters the right hand side of the equation through d . For this example the random-number generator is initialized at the same seed. For Figure 2.2 we again used the open loop optimal control from $[0, T^\infty]$ and applied it to the perturbed Burgers equation. The computing time for Figure 2.2 is 56 units, whereas for the other figures it is 16.3 - 20 units. Thus the ratio of the computing times for the solution on $[0, T^\infty]$ and the receding horizon solutions is less favorable than for 5.1. This is due to the fact that linear systems are less well-conditioned as a consequence of the terminal weight $G \neq 0$. The tracking costs for Figures 2.2 - 2.6 are given in Table 2.

While for Figures 2.2 - 2.4 the noise enters through d , we set $d = 0$ for the results of Figures 2.5 and 2.6. For Figures 2.5 and 2.6 we have noise in the data with $\delta = 10$, and in case of Figure 2.6 also in the initial condition with $\delta = 5$. We also tested with smaller observation domain $\hat{\Omega}$ and obtained comparable results. Reducing the control domain $\tilde{\Omega}$ results in a significant

Fig 2.2	Fig 2.3	Fig 2.4	Fig 2.5	Fig 2.6
.179	.0816	.0814	.0812	.0887

Table 2: \hat{J}

increase of \hat{J} . For example, for the settings of Figure 2.4, if $\tilde{\Omega} = (0.8)$, then $\hat{I} = .490$.

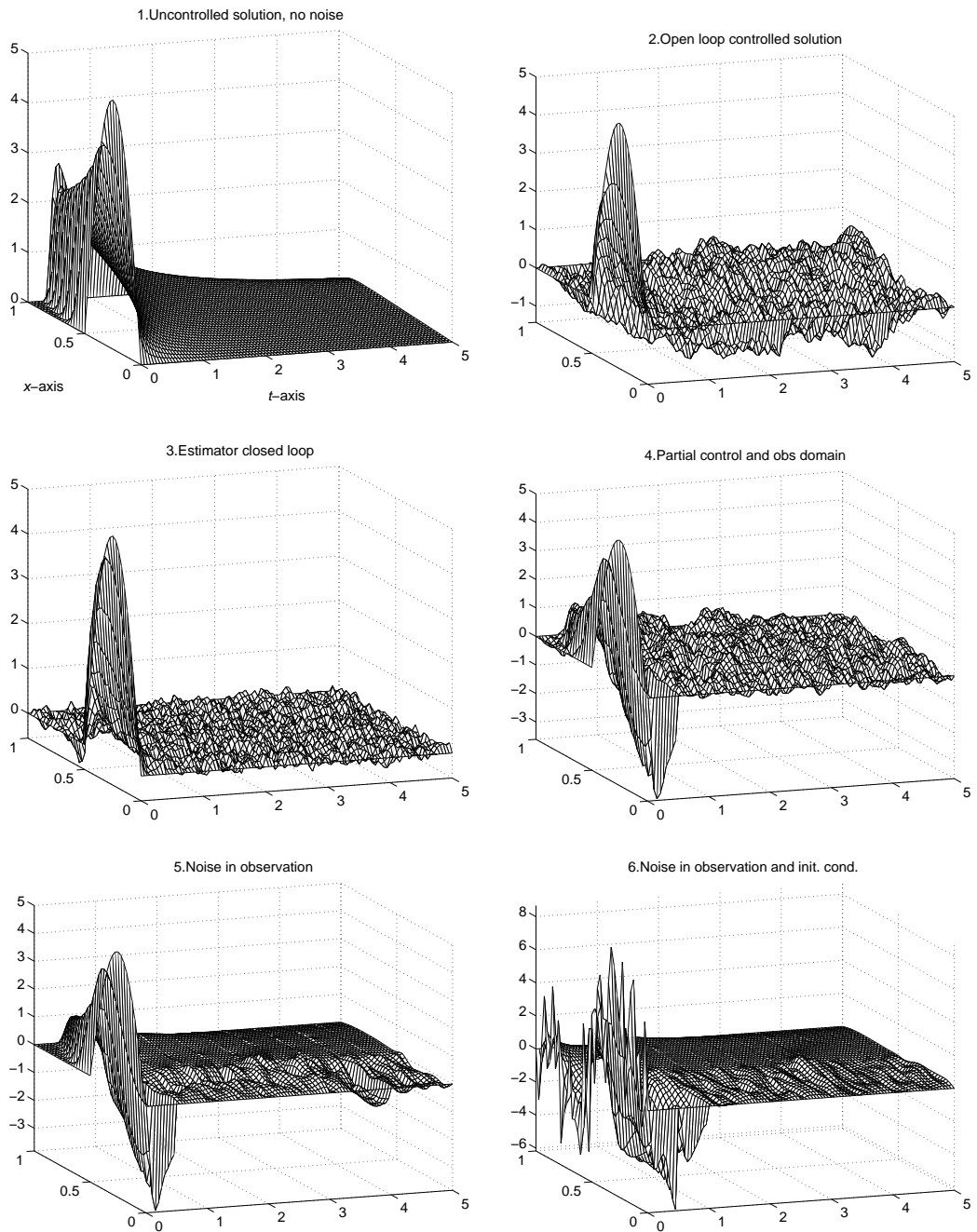


Figure 1

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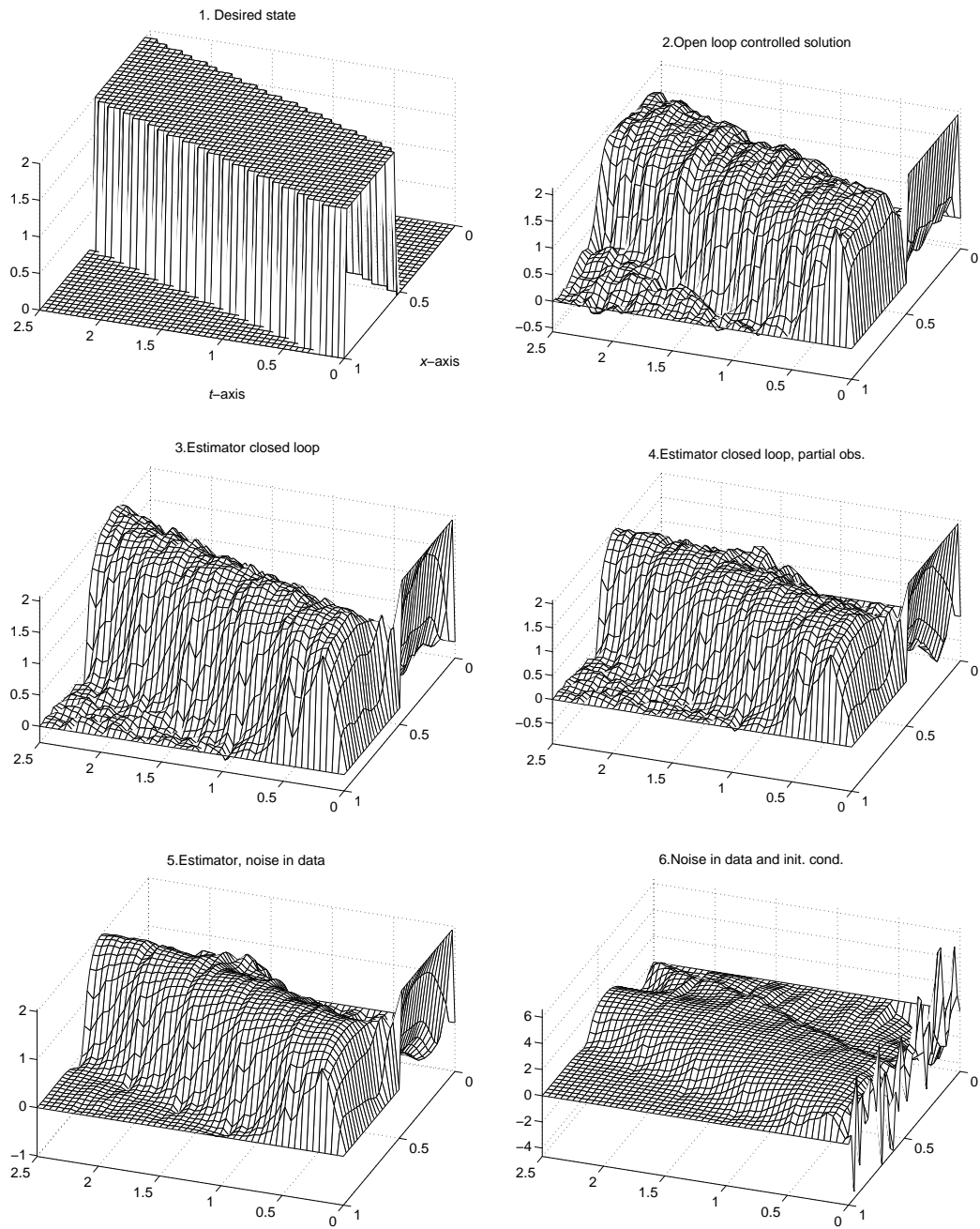


Figure 2

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